Sum rules for total hadronic widths of light mesons and rectilinear stitch of the masses on the complex plane

M. Majewski[a]

Department of Theoretical Physics, University of Lodz, Pomorska 149/153, 90-236 Lodz, Poland

Abstract. Mass formulae for light meson multiplets derived by means of the exotic commutator technique are written for complex masses and considered as complex mass sum rules (CMSR). The real parts of the CMSR give the well known mass formulae for real masses (Gell-Mann–Okubo, Schwinger and ideal mixing ones) and the imaginary parts of CMSR give appropriate sum rules for the total hadronic widths – width sum rules (WSR). Most of the observed meson nonets satisfy the Schwinger mass formula (S nonets). The CMSR predict for the S nonet that the points (m, Γ) form a rectilinear stitch (RS) on the complex mass plane. For low-mass nonets the WSR are strongly violated due to "kinematical" suppression of the particle decays, but the violation decreases as the mass increases and disappears above $\sim 1.5\,\mathrm{GeV}$. The slope k_s of the RS is not predicted, but the data show that it is negative for all S nonets and its numerical values are concentrated in the vicinity of the value -0.5. If k_s is known for a nonet, we can evaluate "kinematical" suppressions of its individual particles. The masses and the widths of the S nonet mesons submit to some rules of ordering which matter in understanding the properties of the nonet. We give the table of the S nonets indicating masses, widths, mass and width orderings. We show also mass–width diagrams for them. We suggest to recognise a few multiplets as degenerate octets. In the appendix we analyze the nonets of the 1^+ mesons.

1 Introduction

The total particle width is one of the main attributes, just as important as the mass and discrete quantum numbers. It tells us something different than the mass and sometimes it may tell more. The widths of particles with similar masses may differ by many orders. Then the widths first inform us which interaction – strong, electromagnetic or weak – is responsible for their decay.

Obviously, the total hadronic widths are not so much differentiated, but still are remarkably various. Within the $SU(3)_f$ meson multiplet the differences are often of the same magnitude as between the masses. Thus they merit attention. However, as the mass formulae have been derived long ago, no relation is known between the total widths. Perhaps such a situation is due to the conjecture that it is not worthy to pay attention to the widths' pattern of the multiplet, as the widths are in a way accidental. Indeed, selection rules and phase space may suppress more or less the decay of particular particle, thus destroying any given regularity. Such an effect should be especially transparent in low-mass multiplets where e.g., for some particles two-body decays are forbidden and many-body decays are suppressed (the meson ω is a striking example). For more massive multiplets, where many decay channels

are opened, we may expect better agreement. However, the prediction may be interesting in any case.

2 Exotic commutator mass sum rules (ECMSR)

The approach is based on the exotic commutator technique [1]. We assume that the following set of exotic commutators vanishes [2]:

$$\left[T_\alpha, \frac{\mathrm{d}^j T_\beta}{\mathrm{d}t^j} \right] = 0 \quad (j = 1, 2, 3, ...), \tag{1}$$

where the T are $SU(3)_f$ generators, t is the time and (α, β) is an exotic combination of indices; that means that T_α, T_β are chosen such that the operator $[T_\alpha, T_\beta]$ does not belong to the octet representation (we use the combination changing strangeness by 2). Putting

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \mathrm{i}[H, T]$$

and using the infinite momentum approximation for the one-particle hamiltonian [3]

$$H = \sqrt{m^2 + p^2} \simeq p^2 + \frac{m^2}{2p} + O\left(\frac{1}{p^2}\right),$$

[a] e-mail: m.majewski@merlin.fic.uni.lodz.pl

we transform (1) into the following system (cf. [2]):

$$[T_\alpha, [\hat{m^2}, T_\beta]] = 0,$$

$$[T_\alpha, [\hat{m^2}, [\hat{m^2}, T_\beta]]] = 0,$$

$$[T_\alpha, [\hat{m^2}, [\hat{m^2}, [\hat{m^2}, T_\beta]]]] = 0, \quad (2)$$

$$\ldots$$

where $\hat{m^2}$ is the squared-mass operator. Taking matrix elements of these operator equations between one-particle octet states and saturating the products of flavoured operators with one-particle intermediate octet states, we get the system of linear equations for the matrix elements $\langle x_8 \mid (\hat{m^2})^j \mid x_8 \rangle$ $(j = 1, 2, 3, ...)$, where $\mid x_8 \rangle$ is the isoscalar octet state. Solving this system, we find

$$\langle x_8 \mid (\hat{m^2})^j \mid x_8 \rangle = \frac{1}{3}a^j + \frac{2}{3}b^j \quad (j = 1, 2, 3, ...). \quad (3)$$

Here a is an isovector particle mass squared and

$$b = 2K - a, \quad (4)$$

with K being the mass squared of the isodublet particle. Notice that

$$\langle x_8 \mid \hat{m^2} \mid x_8 \rangle = \frac{1}{3}a + \frac{2}{3}b = x_8 \quad (5)$$

is the Gell-Mann–Okubo mass.

Let us discuss a nonet. Introduce isoscalar physical states $|x_1\rangle$, $|x_2\rangle$ which are linear combinations of the exact symmetry octet $|x_8\rangle$ and singlet $|x_0\rangle$ states. Consequently, we can write

$$|x_8\rangle = l_1|x_1\rangle + l_2|x_2\rangle, \quad (6)$$

where l_1, l_2 are real and

$$l_1^2 + l_2^2 = 1. \quad (7)$$

Using (6), we transform (3) into the system of equations

$$l_1^2 z_1^j + l_2^2 z_2^j = \frac{1}{3}a^j + \frac{2}{3}b^j \quad (j = 1, 2, 3, ...), \quad (8)$$

where x_1, x_2 are masses squared of the physical isoscalar mesons (we choose $x_1 < x_2$).

Equations (7) and (8) are basic for the further investigation. They are considered as linear equations for the unknown coefficients l_1^2, l_2^2 and they are examined by the gradually enlarging number of equations (8). This enables us to obtain by a uniform procedure the known results of the broken $SU(3)_f$ symmetry. In the succeeding steps, beginning from the first of (8), we find the known mass formulae: Gell-Mann–Okubo (GMO), Schwinger (S) and ideal mixing (IM). We also find the mixing angle in each of these cases. Let us discuss this procedure in more detail.

If we consider only (7) and the first of (8) (one exotic commutator), we find

$$l_1^2 = \frac{x_2 - x_8}{x_2 - x_1}, \quad l_2^2 = \frac{x_8 - x_1}{x_2 - x_1}. \quad (9)$$

Hence, for an octet ($l_1^2 = 1$, $l_2^2 = 0$) we get the GMO mass formula

$$x_1 = x_8. \quad (10)$$

For a nonet we have $l_1^2 > 0, l_2^2 > 0$. Introducing the mixing angle θ^{GMO}

$$l_1^2 = \sin^2 \theta^{\mathrm{GMO}}, \quad (11)$$

we find

$$z_1 \sin^2 \theta^{\mathrm{GMO}} + z_2 \cos^2 \theta^{\mathrm{GMO}} = z_8. \quad (12)$$

This equation determines θ^{GMO} which is usually quoted as the nonet mixing angle. There is no mass formula in this case and θ^{GMO} is determined only by the experimental masses. Such a nonet we call an GMO one.

If we consider (7) and the first two of (8) (two exotic commutators), then we have three equations for two unknown quantities. Solution (9) remains true provided the masses satisfy a consistency condition. This condition is the well known S mass formula. In terms of the parameters a and b the mass formula and the coefficients l_1^2, l_2^2 are

$$(a - x_1)(a - x_2) + 2(b - x_1)(b - x_2) = 0, \quad (13)$$

$$l_1^2 = \frac{1}{3}\frac{(x_2 - a) + 2(x_2 - b)}{x_2 - x_1}, \quad (14)$$

$$l_2^2 = \frac{1}{3}\frac{(a - x_1) + 2(b - x_1)}{x_2 - x_1}. \quad (15)$$

These equations define another mixing angle, the θ^{Sch} one. Such a nonet we call an S one.

If we consider three equations of (8) (three exotic comutators), then we get one more consistency condition:

$$a(a - x_1)(a - x_2) + 2b(b - x_1)(b - x_2) = 0, \quad (16)$$

and consequently we obtain the IM nonet:

$$z_1 = a, \quad z_2 = b \quad \text{and} \quad l_1^2 = \frac{1}{3}, \quad l_2^2 = \frac{2}{3}. \quad (17)$$

The signs of l_1, l_2 are not determined. Ideally mixed isoscalar states arise if $l_1 l_2 < 0$.

It is now obvious that any additional copy of (8) (for $j = 4, 5, ...$) is an identity.

The exotic commutator approach is the *only* known way to derive the ideally mixed states from something else. Alternatively they are postulated.

The current shape of the nonet mass formula

$$(x_2 - x_8)(x_8 - x_1) = \gamma^2 \quad (18)$$

is obtained by diagonalizing the mass matrix

$$\begin{bmatrix} x_0 & \gamma \\ \gamma & x_8 \end{bmatrix},$$

and eliminating x_0, which is the mass squared of the $SU(3)_f$ singlet. The GMO mass x_8 is known for all cases – GMO, S and IM nonets. The mixing parameter γ is undetermined for the GMO nonet, but it is known for the S

nonet. It can be calculated from the mass formula (13) by the substitutions

$$a = x_8 - \frac{2}{3}(b - a), \qquad (19)$$

$$b = x_8 + \frac{1}{3}(b - a), \qquad (20)$$

and the observation that

$$b - a = 2(K - a). \qquad (21)$$

The calculation confirms the "Schwinger Ansatz":

$$\gamma^2 = \frac{8}{9}(K - a)^2. \qquad (22)$$

Data show (or at least suggest) that, *with the one exception of pseudoscalar mesons, all* nonets comply with this expression. So they are the S nonets.

The pseudoscalar mesons π, K, η, η' form the only recognized GMO nonet. Its experimental masses are described by (18) with mixing parameter

$$\gamma^2 = \frac{2}{9}(K - a)^2. \qquad (23)$$

The S nonets are not much different from the IM ones, but none of them is strictly ideal (see Table 1 for an explicit comparison).

For the IM nonet not only γ^2 is determined (see (22)), but also x_0 can be calculated:

$$x_0 = \frac{1}{3}(2a + b). \qquad (24)$$

3 Sum rules for complex masses (CMSR)

3.1 Complex mass operator

We assume that (8) may be considered for complex mass squared. We choose the complex mass operator in the form

$$\hat{m}_c^2 = \hat{m}^2 - \mathrm{i}\hat{m}\hat{\Gamma}, \qquad (25)$$

where $\hat{m}$ and $\hat{\Gamma}$ are hermitean and commute. This operator can be diagonalised and has orthogonal eigenfunctions. That follows from the observation that the operators

$$\frac{1}{2}(\hat{m}_c^2 + \hat{m}_c^{2^+}) = \hat{m}^2 \quad \text{and} \quad \frac{\mathrm{i}}{2}(\hat{m}_c^2 - \hat{m}_c^{2^+}) = \hat{m}\hat{\Gamma} \quad (26)$$

are hermitean and commute.

We use the notation

$$a_c = a - \mathrm{i}\alpha, \quad K_c = K - \mathrm{i}\kappa, \quad z_1 = x_1 - \mathrm{i}y_1, \quad z_2 = x_2 - \mathrm{i}y_2 \qquad (27)$$

for the complex masses of the physical particles, where

$$a = m_a^2, \quad K = m_K^2, \quad x_1 = m_1^2, \quad x_2 = m_2^2, \qquad (28)$$

and

$$\alpha = m_a\Gamma_a, \quad \kappa = m_K\Gamma_K, \quad y_1 = m_1\Gamma_1, \quad y_2 = m_2\Gamma_2. \qquad (29)$$

For subsidiary states with complex masses

$$z_8 = \frac{1}{3}a_c + \frac{2}{3}b_c, \quad b_c = 2K_c - a_c \qquad (30)$$

we use a similar notation:

$$z_8 = x_8 - \mathrm{i}y_8, \quad b_c = b - \mathrm{i}\beta. \qquad (31)$$

The parameters α, κ, y_1, y_2 are positive, as they refer to real particles. Below it will be seen that y_8 is also positive. This concerns the observed values as well as the predicted ones. The question about positivity of β is not so simple. It will be seen further that the predicted value of β is positive for all known S nonets, but as

$$\beta = 2\kappa - \alpha, \qquad (32)$$

it may happen that $\beta < 0$ for observed values of α and κ. That will indicate a relative suppression of the K-meson decay.

It is convenient to introduce the subsidiary widths Γ_8 and Γ_b:

$$y_8 = m_8\Gamma_8, \quad \beta = m_b\Gamma_b. \qquad (33)$$

For complex masses also the coefficients l_1, l_2 (6) are complex and in (7) and (8) the l^2s are replaced by the $\mid l \mid^2$s.

Below we show that the real parts of the masses (25) obey the usual mass formulae. For the imaginary parts there arise some new relations. We call them "width sum rules" (WSR), invoking the name "mass sum rules" (MSR) used sometimes for the mass formulae.

3.2 Gell-Mann–Okubo nonet

From (7) and the first of (8) (for $j = 1$) we find

$$\mid l_1 \mid^2 = \frac{x_2 - x_8}{x_2 - x_1}, \quad \mid l_2 \mid^2 = \frac{x_8 - x_1}{x_2 - x_1}, \qquad (34)$$

and

$$\mid l_1 \mid^2 y_1 + \mid l_2 \mid^2 y_2 = y_8. \qquad (35)$$

$\mid l_i \mid^2$ determines the mixing angle which depends only on the masses and is not affected by the widths. Equation (35) shows that y_8 is positive.

It is seen from (34) that

$$x_1 < x_8 < x_2. \qquad (36)$$

Therefore,

$$\tilde{x}_1 = x_8 - x_1, \quad \tilde{x}_2 = x_2 - x_8 \qquad (37)$$

are positive. Introducing also

$$\tilde{y}_1 = y_8 - y_1, \quad \tilde{y}_2 = y_2 - y_8, \qquad (38)$$

we can write (35) in the form

$$\frac{\tilde{y}_2}{\tilde{x}_2} = \frac{\tilde{y}_1}{\tilde{x}_1}. \qquad (39)$$

Table 1. Schwinger nonets of mesons (m and Γ in MeV). Nonets, ordered by increasing m_K, are described in three rows containing: masses; widths; mixing angle, mass and width ordering. The subscripts a, K, 1, 2 denote isotriplet, isodublet and isoscalar states. $m_b=\sqrt{b}$ and $\Gamma_b=\frac{\beta}{m_b}$ are calculated. In the ordering rules a, b, x_1, x_2 are masses squared and α, β, y_1, y_2 are products of mass and width. Mixing angle θ is the θ^{GMO}, except of 1^{++}, 1^{+-} where it is θ^{Sch}. K_A, K_B are unphysical nonet states (see the appendix). Notation and data are quoted from RPP [4]

J^{PC} particles	m_K Γ_K θ^{GMO}	m_a Γ_a	m_1 Γ_1	m_b Γ_b	m_2 Γ_2
			mass ordering		width ordering
1^{--}	893.88 ± 0.26	769.3 ± 0.8	782.57 ± 0.12	1001.7 ± 1.1	1019.456 ± 0.020
$\bullet\rho(770)$					
$\bullet K^*(892)$	50.7 ± 0.8	149.2 ± 0.7	8.44 ± 0.09	-24.8 ± 2.1	4.26 ± 0.05
$\bullet\omega(782)$					
$\bullet\Phi(1020)$	$(39.28 \pm 0.16)^\circ$	$a < x_1 < b < x_2$		$\alpha > y_1 > \beta > y_2$	
1^{+-}	1324 ± 8	1229.5 ± 3.2	1170 ± 20	1414 ± 9	1386 ± 19
$\bullet b_1(1235)$					
$\bullet K_B$	135 ± 17	142 ± 9	360 ± 40	130 ± 40	91 ± 30
$\bullet h_1(1170)$					
$\bullet h_1(1380)$	$0 \div 35.26^\circ$	$x_1 < a < x_2 < b$		$y_1 > \alpha > y_2 > \beta$	
1^{++}	1340 ± 8	1230 ± 40	1281.9 ± 0.6	1420 ± 12	1426.3 ± 1.1
$\bullet a_1(1260)$					
$\bullet K_A$	134 ± 16	$250 \div 600$	24.0 ± 1.2	$-447 \div 89$	55.5 ± 2.9
$\bullet f_1(1285)$					
$\bullet f_1(1420)$	$35.26^\circ \div 41.00^\circ$	$a < x_1 < b < x_2$		$\alpha > y_1 > \beta > y_2$	
0^{++}	1412 ± 6	984.7 ± 1.2	980 ± 10	1737 ± 11	1713 ± 6
$\bullet a_0(980)$					
$\bullet K_0(1430)$	294 ± 23	$50 \div 100$	$40 \div 100$	$380 \div 490$	125 ± 10
$\bullet f_0(980)$					
$\bullet f_0(1710)$	$(33.5 \pm 2.0)^\circ$	$x_1 < a < x_2 < b$		$y_1 > \alpha > y_2 > \beta$	
2^{++}	1429.0 ± 1.4	1318.0 ± 0.6	1275.4 ± 1.2	1532.0 ± 3.1	1525 ± 5
$\bullet a_2(1320)$					
$\bullet K_2^*(1430)$	103.8 ± 4.0	107 ± 5	$185.1^{+3.4}_{-2.6}$	101.5 ± 11.9	76 ± 10
$\bullet f_2(1270)$					
$\bullet f_2'(1525)$	$(30.67^{+1.56}_{-1.72})^\circ$	$x_1 < a < x_2 < b$		$y_1 > \alpha > y_2 > \beta$	
2^{-+}	1773 ± 8	1670 ± 20	1617 ± 5	1870 ± 33	1842 ± 8
$\bullet\pi_2(1670)$					
$\bullet K_2(1770)$	186 ± 14	259 ± 10	181 ± 11	122 ± 49	225 ± 14
$\eta_2(1645)$					
$\eta_2(1870)$	$0 \div 35.26^\circ$	$x_1 < a < x_2 < b$		$y_1 > \alpha > y_2 > \beta$	
3^{--}	1776 ± 7	1691 ± 5	1667 ± 4	1857 ± 11	1854 ± 7
$\bullet\rho_3(1690)$					
$\bullet K_3^*(1780)$	159 ± 21	161 ± 10	168 ± 10	158 ± 53	87^{+28}_{-23}
$\bullet\omega_3(1670)$					
$\bullet\Phi_3(1850)$	$(32.0^{+3.3}_{-7.5})^\circ$	$x_1 < a < x_2 < b$		$y_1 > \alpha > y_2 > \beta$	

3.3 Schwinger nonet

Consider (7) and the first two of (8) (for $j = 1, 2$). Using for the complex masses (18) with γ^2 given by (22), we find for its real part

$$\tilde{x}_1\tilde{x}_2 - \tilde{y}_1\tilde{y}_2 = \frac{2}{9}[(b-a)^2 - (\beta-\alpha)^2], \qquad (40)$$

and for the imaginary one

$$\tilde{x}_1\tilde{y}_2 + \tilde{x}_2\tilde{y}_1 = \frac{4}{9}(b-a)(\beta-\alpha). \qquad (41)$$

From (39) and (41) we find

$$\tilde{y}_1 = \frac{2}{9}\frac{(b-a)(\beta-\alpha)}{\tilde{x}_2}, \qquad (42)$$

$$\tilde{y}_2 = \frac{2}{9}\frac{(b-a)(\beta-\alpha)}{\tilde{x}_1}. \qquad (43)$$

Substituting (42) and (43) into (40) we get a quadratic equation for the product $\tilde{x}_1 \tilde{x}_2$:

$$(\tilde{x}_1 \tilde{x}_2)^2 - \frac{2}{9} \left[(b-a)^2 - (\beta - \alpha)^2 \right] (\tilde{x}_1 \tilde{x}_2)$$
$$- \left(\frac{2}{9} \right)^2 (b-a)^2 (\beta - \alpha)^2 = 0. \tag{44}$$

The solution

$$\tilde{x}_1 \tilde{x}_2 = \frac{2}{9} (b-a)^2 \tag{45}$$

is just the Schwinger mass formula. It can be written in the form (13). The widths of the particles do not contribute to the mass formula. The second root $\tilde{x}_1 \tilde{x}_2 = -\frac{2}{9}(\beta - \alpha)^2$ is rejected as negative.

As $\tilde{x}_1, \tilde{x}_2$ and $(b-a)$ are positive, it follows from (42) and (43) that $\tilde{y}_1, \tilde{y}_2$ and $\beta - \alpha$ have the same sign. So, y_1 and y_2 lie on opposite sides of y_8.

Multiplying (42) by (43) and using (45), we find the formula

$$\tilde{y}_1 \tilde{y}_2 = \frac{2}{9} (\beta - \alpha)^2, \tag{46}$$

which is the analogue of the Schwinger mass formula.

Finally, we notice that (cf. (21))

$$\beta - \alpha = 2(\kappa - \alpha) \tag{47}$$

and that WSR (46) can be written in the form (cf. (13))

$$(\alpha - y_1)(\alpha - y_2) + 2(\beta - y_1)(\beta - y_2) = 0. \tag{48}$$

3.4 Ideal nonet

If we use (7) and the first three of (8) (for $j = 1, 2, 3$), we obtain (17) for complex masses

$$z_1 = a_c, \quad z_2 = b_c, \quad \text{and} \quad |l_1|^2 = \frac{1}{3}, \quad |l_2|^2 = \frac{2}{3}. \tag{49}$$

Hence,

$$x_1 = a, \quad x_2 = b, \quad \Gamma_1 = \Gamma_a, \quad \Gamma_2 = \Gamma_b. \tag{50}$$

Note that (7) and the three equations of (8) give four real conditions and three imaginary ones. The first two of the real conditions determine $|l_1|^2$, $|l_2|^2$; the remaining two give ideal values for x_1, x_2. For calculating Γ_1, Γ_2 we have three imaginary equations. However, for ideal masses the ideal Γ_1, Γ_2 follow from the first two equations of (8) and the third equation does not change the result.

4 The rectilinear stitch of the masses on the complex plane

The formulae (34), (45) and (49) for the real parts of the complex masses are identical with the corresponding formulae for the real ones. So, the conditions of flavour-symmetry breaking which operate in CMSR well reproduce the data on the masses.

We assumed from the beginning that also in the WSR (35), (46) and (49), flavour-symmetry breaking factors are correctly taken into account. However, this does not mean that they have to describe well the real data. The particle widths depend also on non-flavour factors which violate the WSR. Let us call them for brevity "kinematical", although they may include other effects. Among the "kinematical" factors the main role is played by strictly kinematical ones – the phase-space volume and conservation laws. These factors may considerably disturb the widths of the low-mass particles, but the higher is the mass, the weaker is their influence due to opening of new decay channels. Therefore, violation of the WSR would be significant in low-mass nonets and would weaken for more massive ones. The data reveal such a tendency. It is, therefore, likely that our initial assumption is true (the flavour-symmetry breaking is really well described). Then violation of the WSR is a measure of the "kinematical" violation. However, it is difficult to evaluate on the basis of WSR the size of the violation and attribute it to definite particles of the nonet, because the sum rules do not include the "triangulation point". One can make the "kinematical" breaking transparent combining WSR with the mass formula.

Equations (42) and (43) connect the real and imaginary parts of the masses of the S nonet. By means of (45), we obtain the very simple result (cf. (39)):

$$\frac{\tilde{y}_2}{\tilde{x}_2} = \frac{\tilde{y}_1}{\tilde{x}_1} = \frac{\beta - \alpha}{b - a} = k_s. \tag{51}$$

The points (a, α), (K, κ), (x_1, y_1), (x_2, y_2), (x_8, y_8), (b, β) lie on a *straight line* in the plane $(m^2, m\Gamma)$. Also the points (m_a, Γ_a), (m_K, Γ_K) etc. lie on a straight line with the *same* slope in the plane (m, Γ). The slope k_s is indefinite.

So, the masses form a rectilinear stitch (RS) on the complex plane.

Equations (51) ultimately explain the nature of violation of the WSR of the S nonet. If the slope of the stitch $k_s \neq 0$ (true for all S nonets), then the WSR and the S mass formula are equivalent and satisfy the same conditions of the broken flavour symmetry. Therefore, flavour breaking does not operate and violation of the WSR of the S nonet is "kinematical".

Reversing the argument, one can say that rectilinearity of the stitch is a result of flavour-symmetry breaking which identically influences the mass formula and the WSR.

5 Ordering rules

The expressions (14) and (15) derived for an S nonet do not guarantee positivity of $|l_1|^2$, $|l_2|^2$. This is required additionally. The requirement introduces further restrictions on the masses which take the form of an ordering rule. There are two allowed mass orderings of the S nonet (cf. [6]) implying also two distinct inequalities between some masses and two different ranges of the mixing angle θ^{Sch}:

$$a < x_1 < b < x_2; \quad 2K < x_1 + x_2; \quad \theta^{\text{Sch}} > \theta^{\text{id}}, \quad (52)$$

$$x_1 < a < x_2 < b; \quad 2K > x_1 + x_2; \quad \theta^{\text{Sch}} < \theta^{\text{id}}, \quad (53)$$

where $\theta^{\text{id}} = 35.26°$ is the ideal mixing angle. The mass ordering rules follow from (13), (14) and (15) under the conditions $\mid l_1 \mid^2 > 0$, $\mid l_2 \mid^2 > 0$. The inequalities for the masses follow from the ordering and the relation $a + b = 2K$. The inequalities for θ^{Sch} follow from the ordering and the equation

$$\tan^2 \theta^{\text{Sch}} = \frac{\mid l_1 \mid^2}{\mid l_2 \mid^2}. \quad (54)$$

To obtain the ordering rules for the widths we combine

$$y_8 = \alpha + \frac{2}{3}(\beta - \alpha), \quad (55)$$

$$y_8 = \beta - \frac{1}{3}(\beta - \alpha), \quad (56)$$

with (42) and (43) and observe that for the nonet (52)

$$\tilde{x}_1 < \frac{2}{3}(b - a), \quad \tilde{x}_2 > \frac{1}{3}(b - a) \quad (57)$$

and for the nonet (53)

$$\tilde{x}_1 > \frac{2}{3}(b - a), \quad \tilde{x}_2 < \frac{1}{3}(b - a). \quad (58)$$

We thus find two possible width orderings for each mass ordering: for the rule (52) we find

$$\alpha < y_1 < \beta < y_2 \quad \text{or} \quad \alpha > y_1 > \beta > y_2 \quad (59)$$

and for the rule (53)

$$y_1 < \alpha < y_2 < \beta \quad \text{or} \quad y_1 > \alpha > y_2 > \beta. \quad (60)$$

For the GMO nonet the only restriction on the masses (36) follows from the requirement of positivity of the $\mid l \mid^2$. Therefore, besides the rules (52) and (53) there are also the possible inequalities

$$a < x_1 < x_2 < b \quad \text{and} \quad x_1 < a < b < x_2, \quad (61)$$

where the conditions (52) and (53) for K and θ^{GMO} do not hold. In particular, the equality $\theta^{\text{GMO}} = \theta^{\text{id}}$ is possible for a nonideal nonet: $x_1 \neq a$, $x_2 \neq b$. Therefore, for such a nonet the value of θ^{GMO} would not yield a criterion of ideality. However, we do not know such a nonet as of yet.

6 Bird's eye view on the nonet data

Table 1 collects data on seven S nonets ordered by increasing K. To make the data more transparent the physical masses and widths quoted from PDG [4] are supplemented with the calculated values of $m_b = \sqrt{b}$ and $\Gamma_b = \frac{\beta}{m_b}$ ("mass" and "width" of the ideal state $s\bar{s}$). We also indicate for each nonet the mixing angle θ as well as the mass and width ordering.

In the IM nonet the numbers from neighbouring columns 3 and 4 as well as 5 and 6 would be equal. As they are not, the nonets are not ideal. Instead, all of them are S nonets. That can be checked by saturating the Schwinger mass sum rule with the masses lying within the bounds of experimental error. These masses define the mixing angle θ^{Sch}. On the other hand, we can calculate θ^{GMO} using mean experimental values of the masses. The mixing angle θ (we assume $0 < \theta < \frac{\pi}{2}$) quoted in Table 1 is in most cases the θ^{GMO} one. The calculated θ often have big errors. In several cases we cut the errors using restriction (52) or (53). Observe that for the orderings (52) or (53) which are allowed by the Schwinger mass formula, the requirements formulated for θ^{Sch} are also valid for θ^{GMO} and that these angles are not far removed from each other in the vicinity of θ^{id}.

Some further remarks are in order.

Two known pseudoscalar multiplets are not included in Table 1. The first is the nonet π, K, η, η' which is not the S one and, besides, has no hadronic decays (except of η'). The second is the multiplet (nonet?) $\pi(1300)$, $K(1460)$, $\eta(1295)$, $\eta(1440)$ for which we cannot establish even the mass ordering, owing to big errors of the $\pi(1300)$ and $K(1460)$ masses.

The masses of the unphysical states K_A (1^{++}) and K_B (1^{+-}) are required to satisfy the S nonet constraints. That makes their mixing angles the θ^{Sch} ones by definition. The states K_A and K_B are superpositions of the physical states $K_1(1270)$, $K_1(1400)$ and therefore the masses of K_A and K_B must satisfy an additional condition imposed by mixing. These three constraints prove to be very restrictive, and we find that the values of K_A and K_B obeying them are contained within narrow intervals which are comparable with the error ranges of the $K_1(1270)$ and $K_1(1400)$ masses. Also we find that the values of the a_1-meson mass allowed by these constraints cover only part of the range of the experimental data. For the details of the procedure, see the appendix.

The values of β (see (32)) for the nonets 1^{--} and 1^{++}, calculated from the data on the a- and K-meson widths, come down deeply into the region of negative values (Table 1), while the sum rules predict $\beta > y_2$. In both cases we accept the width of the a-meson as a measure of the "normal" (unsuppressed) width suitable for the nonet (even if the error of the a_1-meson width is so big). In the 1^{--} nonet, $\beta < 0$ explicitly indicates a deficiency of the K^*-meson width. In the nonet 1^{++}, the scope of the calculated β comprise negative values as well as positive ones. The negative values may be explained by the big error of the a_1 width, without invoking a deficit of the K_A width. In that case (if the width of K_A were really not reduced) the a_1 width would be close to the lower limit of the experimental value. Similar remarks can also be made for other multiplets. However, that and other disagreements between prediction and data are better seen from mass–width diagrams.

Figure 1 exhibits mass–width diagrams of the same S nonets. On each diagram, besides the points (m, Γ) representing the observed mesons, we draw the straight line
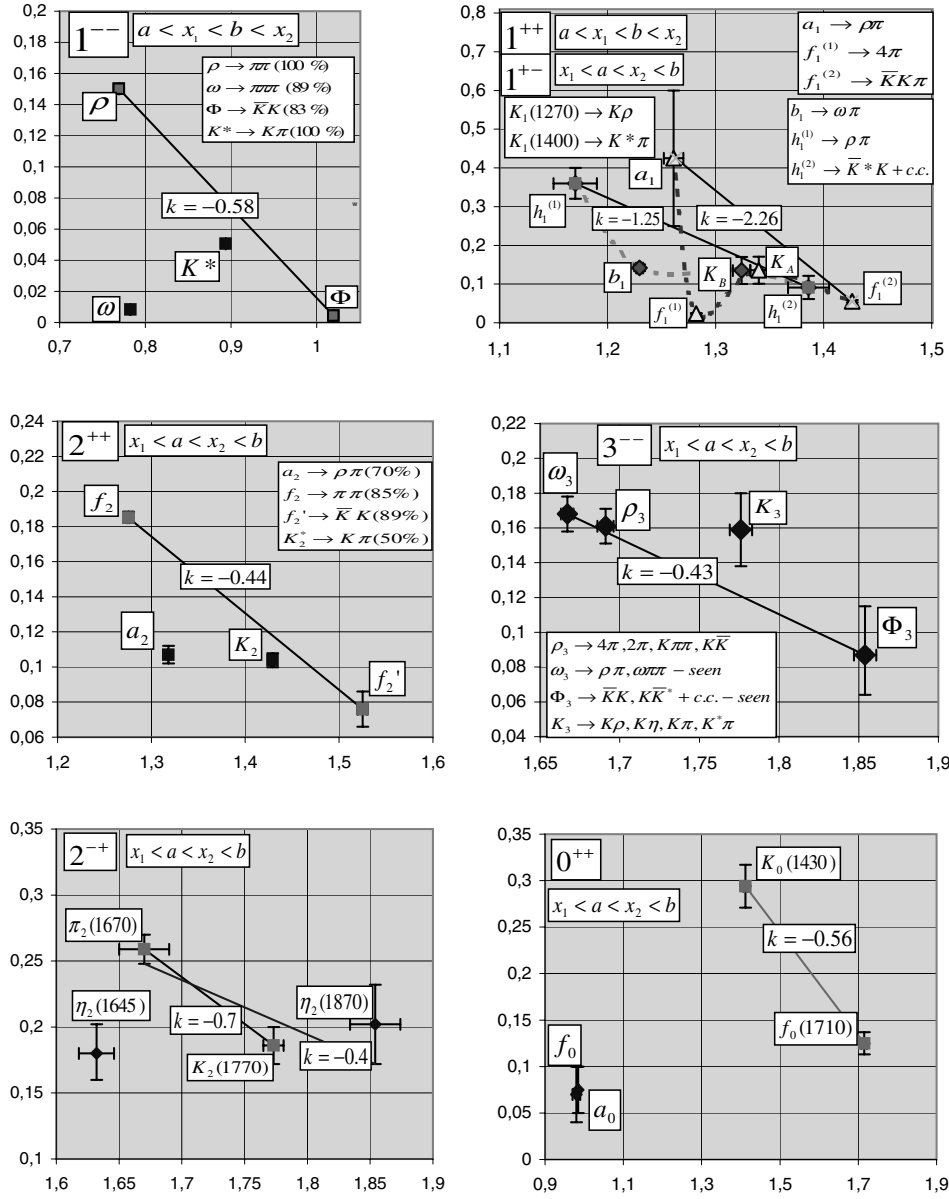
Fig. 1. Mass–width diagrams of the S nonets. On the axes m and Γ are in GeV. Shown are the dominating decays of some particles. Full straight lines crossing two or more points are supposed to be nearest by RS of the nonet. For 3^{--} and 2^{++} they are expected to lie along the RS. For 1^{--} the small phase space pushes down the observed point Φ. For the 1^+ nonets the straight lines cannot be identified with RS

crossing two or more of the points. With the one exception of $K_3(1780)$ (the data on $\eta_2(1870)$ we consider uncertain), the line is drawn in such a way that there are no experimental points lying above it. That follows from the guess that deviations from the RS occur only downward. The guess itself reflects the observation that suppressing-decay mechanisms are well known and frequent, while nothing certain is known about enhancing-decay mechanisms. For some of the nonets we also show dominating channels of the decay. We wonder whether the straight lines can be identified with the RS.

The most striking feature of these diagrams is the negative slope of the straight lines for all nonets (heavier particles have smaller width).

Let us discuss the diagrams of some nonets in more detail. Begin from the nonets 1^{--}, 2^{++} where we have the most complete data. We draw the straight lines over the points (ρ, Φ) for 1^{--} and (f_2, f_2') for 2^{++}. The dominating channels of the hadronic decays of these particles $\rho \to \pi\pi$, $\Phi \to K\bar{K}$, $f_2 \to \pi\pi$, $f_2' \to K\bar{K}$ are J, P, C, G, S-allowed, but in the case of $\Phi \to K\bar{K}$ the phase space is small and the width would be relatively reduced. Therefore, if we want to identify these lines with the RS, we should expect the 1^{--} line to be steeper. From Fig. 1 we read off $k_s(1^{--}) = -0.56$ and $k_s(2^{++}) = -0.44$.

For the nonet 3^{--} we draw the line over the points ω_3, ρ_3, Φ_3. Each of the particles has a phase space large enough for many decay channels. Apparently, at these energies the number of opened channels is sufficient for the particles to satisfy the RS equation. From Fig. 1 we read off $k_s(3^{--})=-0.43$.

It is, therefore, likely that the slopes of RS are similar for the nonets 1^{--}, 2^{++}, 3^{--} and are concentrated inside the interval $k_s = -0.5 \div -0.4$.

The diagrams of the 2^{-+} and 0^{++} nonets do not conflict with this observation. We do not appeal to them, as

they suffer from uncertain data on the isoscalar mesons (2^{-+}) or from an uncertain nonet assignment (0^{++}).

The situation is different with 1^+ nonets. For the mesons of both 1^{+-} and 1^{++} nonets the decays into $\pi\pi$ and $K\bar{K}$ are kinematically forbidden. Their two-particle decays producing heavy-meson (ρ, ω, etc.) and many-particle decays are mostly more or less suppressed by the phase space. For the 1^{++} nonet where we have more data, the straight line shown on Fig. 1 crosses the points a_1 and $f_1(1420)$. Perhaps the decays of the a_1-meson ($a_1 \rightarrow (\pi\rho)_{\text{S wave}}$ etc.) may be considered unsuppressed, as this particle has many decay channels opened and a huge width (although the error is extremely big); but the decays $f_1(1420) \rightarrow K\bar{K}\pi, K\bar{K}^*(892)+$c.c. are clearly suppressed. So the line is based on the particle which has kinematically suppressed decays. Therefore, we cannot accept it as the RS one. A similar situation holds for the 1^{+-} nonet.

7 Degenerate octet

The ninth meson does not mix with the octet, if $l_2^2 = 0$. Then, $x_1 = x_8$, $l_1^2 = 1$, and x_2 is arbitrary. It follows also from (45) that $b = a$ and thus $K = a = x_8$. Then, from (42) we find $\tilde{y}_1 = 0$, from (46) $\beta = \alpha$ and thus $\beta = \kappa = y_8$. We see that the restrictions put by the first two exotic commutators (1) on the octet states provide their full degeneration. This result remains unchanged, if further exotic commutators are included.

Table 2 shows two 1^{--} multiplets and the 4^{++} multiplet which can be understood as degenerate octets. The degeneracy of these octets is clearly seen from the data on the masses. It is confirmed by data on the widths.

On the other hand the data on the widths confirm the suggestion that at these energies the WSR are well satisfied.

8 Summary and discussion

The descripton of unstable particle quantum states via attributing to them a complex mass has been under study from various points of view for a long time (see, e.g., [5] and references therein). We make an attempt, for the first time, to apply the idea of a complex mass for the derivation of extended mass formulae within the early proposed algebraic approach [2] based on exotic commutators of the $SU(3)_f$ charge operators and their time derivatives in the infinite momentum frame system. The new sum rules including hadronic widths of the resonances are compared with the data for ten meson multiplets and the gains and problems encountered are of a certain interest and may also serve as a starting point for a further investigation.

The real parts of the complex mass sum rules (CMSR) give relations between the real parts of the complex mass squared which are identical with the well known mass formulae of the broken flavour symmetry for real masses.

The CMSR predict two possible kinds of octet: Gell-Mann–Okubo (with masses satisfying the GMO mass formula) and degenerate. So, the latter is obtained not only

Table 2. Degenerate octets (m and Γ in MeV [4])

J^{PC}	m_a Γ_a	m_K Γ_K	m_1 Γ_1	
1^{--}				
•$\rho(1450)$	1465 ± 25	1414 ± 15	1419 ± 31	
•$K^*(1410)$				
•$\omega(1420)$	310 ± 60	232 ± 21	174 ± 60	
1^{--}				
•$\rho(1700)$	1700 ± 20	1717 ± 27	1649 ± 24,	1680 ± 20
•$K^*(1680)$				
•$\omega(1650)$	240 ± 60	322 ± 110	220 ± 35,	150 ± 50
•$\Phi(1680)$				
4^{++}				
•$a_4(2040)$	2011 ± 13	2045 ± 9	2025 ± 8	
•$K_4(2045)$				
•$f_4(2050)$	360 ± 40	198 ± 30	194 ± 13	

for exact symmetry but also under the CMSR constraints. The CMSR predict width degeneration of the mass degenerate octet particles. The data shown in Table 2 demonstrate that two 1^{--} multiplets and the 4^{++} one are degenerate octets. The degeneration is seen not only from the masses, but also from the widths.

The CMSR predict three kinds of nonets: Gell-Mann–Okubo, Schwinger (S) and ideally mixed (IM). All the observed nonets having hadronic widths are S ones. As the S nonet well describes the masses, we may think that it correctly takes into account the flavour breaking factors.

The S nonet mesons are submitted to definite mass ordering. There are two allowed orderings: $x_1 < a < x_2 < b$ and $a < x_1 < b < x_2$ (for the notation, see Sect. 2). The mass ordering decides whether the mixing angle of the nonet is smaller or bigger than θ^{id}. It also decides whether $2K$ is bigger or smaller than $x_1 + x_2$. Thus the mass ordering is a relation characterizing the nonet; likewise the mixing angle characterises it.

The imaginary parts of the CMSR give relations between the imaginary parts of the complex mass squared. These relations – the width sum rules (WSR) of GMO, S and IM types – connect the total widths of the nonet mesons. WSR have shapes identical with the mass formulae, but, in contrast, they are not satisfied by the data in general. The reason is an extra reduction of the widths of the individual particles by "kinematical" factors (we mean not only strictly kinematical factors, like phase-space and E, I, S, J, P, C, G conservation, but also all other non-flavour ones). This violation diminishes along with the growth of the mean nonet mass and disappears at about 1.5 GeV.

In the S nonets the ordering rules are in force also for the imaginary parts of the masses. They are correlated with the mass ordering rules, but also are not satisfied by the observed widths of the low-mass nonets.

Table 1 collects seven S nonets. For each of them are shown masses, widths, mass and width ordering, and mix-

ing angle. Deviations of the isoscalar mesons from ideal values are apparently seen.

For the S nonet CMSR predict a linear relation between the masses and the widths of all particles. Hence, the complex masses form a rectilinear stitch (RS) on the complex plane with a slope k_s depending of the nonet. As $k_s \neq 0$, the Schwinger WSR is equivalent to the S mass formula and complies with the conditions breaking flavour symmetry. Therefore, violation of the WSR cannot be a result of the flavour-symmetry breaking and consequently *violation of the WSR for the S nonet is "kinematical"*.

To construct the RS of a nonet we need the particles decaying in the "kinematically" unsuppressed processes. We can do that for the nonet having masses above 1.5 GeV where many channels of the decay are opened and WSR are satisfied. 3^{--} is such a nonet. Also for a low-mass nonet the RS can be determined approximately, if there exist two particles with "kinematically" unsuppressed dominating decays. The nonets 1^{--} and 2^{++} are examples. Figure 1 demonstrates mass–width diagrams of seven nonets. For the nonets 1^{--} and 2^{++} the RS is constructed as the highest lying straight line crossing two experimental points (m, Γ) of the diagram. For the 1^+ nonets such a line cannot be identified with the RS, because of lack of two particles decaying in a "kinematically" unsuppressed reaction.

In the cases where k_s is known we can evaluate the "kinematical" suppression of a particle as the difference between predicted (lying on RS) and observed value of the width.

The rectilinearity of the stitch and the stitch itself result from flavour-symmetry breaking (for an exactly symmetric multiplet there would be only one point), but the slope of the stitch k_s is undetermined. The data show that k_s is negative for all nonets. They also suggest that the slopes are not much different from each other and are concentrated somewhere in the region $k_s \approx -0.5 \div -0.4$. The origin of such behaviour of $\Gamma(m)$ is unknown. We can only remark that it resembles the behaviour of the strong coupling constant α_s, also decreasing with increasing mass and being flavour independent.

Let us go back to the idea with which we began this paper. Hadron widths do not influence the mass formula of broken flavour symmetry and the mixing angle of the nonet. Therefore, one could once discover them and describe the nonet. That is the reason why hadron spectroscopy could completely ignore the data on the total widths of the particles. However, hadrons do have finite widths and hadron spectroscopy should obligatorily describe them. Moreover, the widths are considerable and may be important for the description of the multiplet as a whole, not only as attributes of individual particles. But where is a trace of that? CMSR introduce the widths into our scope. They predict not only the WSR connecting the total widths of the nonet mesons, but also the RS being the result of the interplay between real and imaginary parts of the CMSR. That is the place where we can expect something new. The RS is such a relation characterizing

all S nonets; perhaps the slope k_s characterises the nonet individually.

Much attention is devoted nowadays to the existence of glueballs. The search for these states requires detailed information on the multiplets. A glueball cannot be discovered by analysing the properties of the single-particle state only. Even more, such a discovery would not be convincing. A glueball state with non-exotic quantum numbers J^{PC} should mix with isoscalar $q\bar{q}$ states. Therefore, the way to identify the glueball is to look for meson decouplets including three isoscalar physical mesons and investigate them [6]. Sum rules for complex masses may be useful in such analysis. Investigating of the sum rules for decouplets is in progress.

Appendix: 1^+ multiplets as S nonets

Below, the 1^{++} mesons are called a_A, K_A, x_{A1}, x_{A2} (nonet A) and the 1^{+-} mesons are called a_B, K_B, x_{B1}, x_{B2} (nonet B) (cf. Table 1). We assume that each of the nonets satisfies the Schwinger mass formula. As a function of a and K this formula is the equation of an ellipse:

$$3a^2 + 8K^2 - 8aK + a(x_1 + x_2) - 4K(x_1 + x_2) + 3x_1x_2 = 0. \tag{62}$$

The parameters of the ellipse are determined by the masses of the isoscalar mesons x_1, x_2 which fix the position of the ellipse centre:

$$a^{Cr} = K^{Cr} = \frac{1}{2}(x_1 + x_2), \tag{63}$$

and the magnitude of its axes **a**, **b** which are proportional to $(x_2 - x_1)$. However, they do not influence the axes' ratio (**a/b** = 3.6) and the orientation of the ellipse in the plane (the angle beween big axis and obscissa is 29°). So these quantities are the same for all S nonets.

As the straight line $2K = x_1 + x_2$ crosses the centre of the ellipse, the mass ordering (52) or (53) decides whether K lies below or above its diameter.

The physical states of the 1^+ mesons $K_1(1270)$ and $K_1(1400)$ are mixed states of the K_A and K_B:

$$|K_1(1270)\rangle = |K_A\rangle \cos\Phi + |K_B\rangle \sin\Phi,$$
$$|K_1(1400)\rangle = -|K_A\rangle \sin\Phi + |K_B\rangle \cos\Phi, \tag{64}$$

where Φ is the mixing angle [4]. For the masses squared of these mixed states we have

$$K_A + K_B = K_1(1270) + K_1(1400). \tag{65}$$

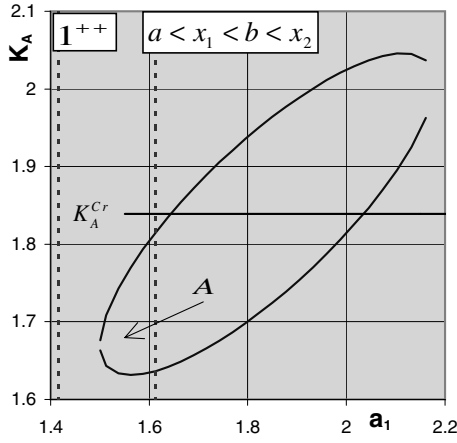We are looking for such values of K_A and K_B which satisfy this equation and (62) for each nonet.

Fig. 2. Ellipse A. The Schwinger mass formula as a function of a_1 and K_A. Dotted lines bound the error corridor of a_1. The allowed K_A belong to segment A of the ellipse. They lie below the diameter K_A^{Cr} as required by mass ordering. On the axes a_1 and K_A are in GeV^2
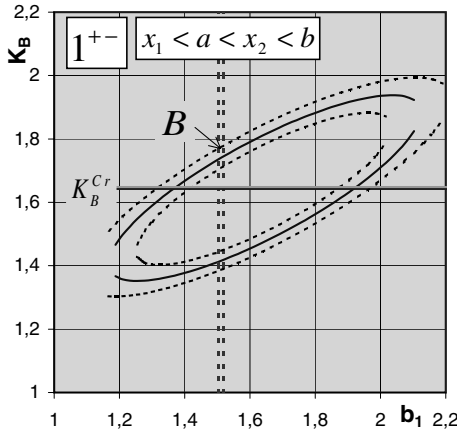

Fig. 3. Ellipse B. The mass errors of $h_1^{(1)}, h_1^{(2)}$ create the corridor shown between the dotted ellipses. Its crossing with the b_1 corridor produces solutions for K_B above and below the ellipse diameter K_B^{Cr}. The choice $K_B > K_B^{Cr}$ follows from mass ordering. On the axes b_1 and K_B are in GeV^2

Ellipse A. The masses of the 1^{++} isoscalar mesons $f_1(1285)$ and $f_1(1420)$ have negligible errors, so the ellipse A is precisely determined. It is shown on Fig. 2. We can see that a_A may be assigned to the S nonet, only if

$$a_A > (1225\,\text{MeV})^2. \qquad (66)$$

This, together with the experimental limit [4], gives

$$(1225\,\text{MeV})^2 < a_A < (1270\,\text{MeV})^2. \qquad (67)$$

From Fig. 2 we find

$$(1277\,\text{MeV})^2 < K_A < (1347\,\text{MeV})^2. \qquad (68)$$

Ellipse B. The mases of the 1^{+-} isoscalar mesons $h_1(1190)$, $h_1(1380)$ have considerable errors which influence the parameters of the ellipse (62). Figure 3 presents

three ellipses corresponding to the mean experimental values of x_{B1}, x_{B2} and to these two combinations of their lower and upper experimental limits which give the ellipses with minimal and maximal axes. Putting for a_B its experimental value, we find two regions of solutions for K_B. The lower solutions are rejected by the mass ordering (53) and we obtain

$$(1307\,\text{MeV})^2 < K_B < (1332\,\text{MeV})^2. \qquad (69)$$

Observe that in spite of the less accurate data on the isoscalar meson masses, the evaluation of the K_B is more accurate than of the K_A one.

Summing (68) and (69) we find

$$(1828\,\text{MeV})^2 < K_A + K_B < (1895\,\text{MeV})^2. \qquad (70)$$

On the other hand, from the data on the masses of the K_1-mesons we have

$$(1884\,\text{MeV})^2 < K_1(1270) + K_1(1400) < (1904\,\text{MeV})^2. \qquad (71)$$

Comparing (71) with (69), we conclude that

$$K_B < \frac{1}{2}(K_1(1270) + K_1(1400)), \qquad (72)$$

and therefore,

$$K_B < K_A. \qquad (73)$$

Comparing (70) with (71), we find that the restrictions on the K_A- and K_B-meson masses following from the nonet assignment are compatible with the observed mass values of K_1-mesons within the narrow interval of the sum K_A+K_B:

$$(1884\,\text{MeV})^2 < K_A + K_B < (1895\,\text{MeV})^2, \qquad (74)$$

where the upper limit of the sum is the sum of the individual upper limits of K_A (68) and K_B (69). Therefore, it immediately follows that the adjustable values of K_A and K_B are

$$K_A = (1340 \pm 8)^2\,\text{MeV}^2, \qquad (75)$$

$$K_B = (1324 \pm 8)^2\,\text{MeV}^2. \qquad (76)$$

Returning to the ellipse A, we find the adjustable value of a_A:

$$a_A = (1261 \pm 9)^2\,\text{MeV}^2. \qquad (77)$$

The mixing angle Φ, which can be calculated from (75) and (76) and the physical masses of the K_1-mesons, is charged with a big error exceeding the difference $(\Phi^{\text{mean}} - 45°)$. Therefore, for evaluating Γ_A and Γ_B (using the physical Γ_Ks) we put $\Phi = 45°$. The calculated widths are indicated in Fig. 1.

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